From Posteriors to Priors via Cycles

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Abstract

We present new necessary and sufficient conditions for checking if certain players' posteriors can be rationalized by a common prior. We propose a simple diagrammatic device to calculate the join and meet of players' knowledge partitions. Each cycle in the diagram has a corresponding cycle equation that must be satisfied. Besides having a geometric interpretation, our conditions differ from those in the literature because they do not use infinities in any sense, not even implicitly or indirectly, and they characterize the set of players' partitions that automatically allow any posteriors to be rationalized by a common prior. This indicates that to assume the existence of a common prior may be a different assumption in different games. We also prove that to assume that posteriors can be rationalized by a common prior is equivalent to assuming that players have the same degree of optimism. We show how to construct a bet (in which it is always common knowledge that all players have positive expected gains) over any cycle whose corresponding equation is not satisfied. A common prior will exist when each player's posterior about her opponents' types is independent of her own type.

Key words: cycle, game, incomplete information, prior, posterior, type.

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Contents

1 Introduction

Economists have been studying the role of information in game theory extensively over the last decades.[1] In the traditional framework of an incomplete information game, players have a common prior distribution over the set of possible states of the world. Then, after observing a private signal, they update their beliefs in a Bayesian fashion. In the final stage, they choose actions and receive the payoffs. Is the common prior assumption equality important in every game? Are we arbitrarily imposing some specific type of behavior by assuming the existence of a common prior? To answer these questions we need to understand what kind of behavior is consistent with the assumption that all posteriors come from a common prior. A step in this direction is the characterization of such posteriors.[2]

We present new necessary and sufficient conditions for checking if a set of posteriors, one for each player, can be rationalized by a common prior. The only potential difficulty in obtaining a common prior is due to the existence of cycles in the diagram we introduce. Each cycle is associated to an equation. We describe these cycle equations and prove that all of them are satisfied if and only if it is possible to rationalize posteriors using a common prior. The cycles provide the link between geometry of knowledge partitions and the algebraic characterization of rationalizability via cycle equations.

Intuitively speaking, the assumption of rationalizability by a common prior is a different requirement in different games. We exemplify this insight in 4 applications. In the first, we consider the class of monotonic models, that is, games of incomplete information in which the state space has a linear order and every player's knowledge partition respects this order, in the sense that every atom of every partition has only consecutive states. Any game of 2 players where one of them has better information than the other would be an example of a monotonic model.

In a second application a Cournot duopoly game with uncertain marginal costs is studied. We reveal the equivalence between assuming that posteriors can be rationalized by a common prior and assuming that players have the same degree of optimism. The third application proves that if each player has beliefs about her opponents' types that are independent of her own type, then all cycle equations hold, and therefore, posteriors may be rationalized by a common prior.[3]

The forth application deals with bets. It is well known that there is no bet for which it is always common knowledge that all rational, risk neutral players have positive expected gains if and only if posteriors may be rationalized by a common prior. Therefore, the cycle equations characterize the non-existence of bets. For each cycle such that the corresponding equation is violated, we provide an algorithm to

[1] For a recent survey on the use of knowledge in economics see Samuelson (2004).

[2] Our definition of posterior differs from some authors. See section 2.2 and the remark there.

[3] The word *type* here has a different meaning than it has in Mertens and Zamir (1985), or Brandenburger and Dekel (1993). See section 5.3 for more details.

construct a bet, over the states in the cycle, in which it is common knowledge that all players have positive expected gains. Furthermore, in this case, there is an arbitrage opportunity, that is, a market maker can intermediate this bet and make a risk free positive profit in every state, even if she has no information about the likelihood of any state of this cycle.

1.1 Literature Review

There are other necessary and sufficient conditions for the existence of a common prior. Samet (1998a) represents possible priors for a player as the convex hull of her types, and presents a separation theorem. He proves that there can be no disagreement in expectation if and only if the posteriors can be rationalized by a common prior. Samet (1998b) explores the Markovian structure of beliefs revision and shows how priors can be expressed in terms of players' posteriors via iterated expectations of random variables. His main result establishes another necessary and sufficient condition for common prior existence: convergence of infinite iterated expectations of random variables. Our approach is based neither on iterated expectations, nor on separation theorems. We present finite algebraic conditions, each one with a finite number of terms. There is no infinite regress. This is possible because of the introduction of the geometric concept of a cycle in our diagram. This approach allows us to characterize all bets over a cycle, an object not present in any of the cited papers. The insight here is that disagreements necessarily come from disagreements over cycles in finite state space models. Moreover, depending on players' partitions, there will be no cycles, and then, a common prior will automatically exist.[4]

Many authors studied no trade theorems and disagreement results. This includes Milgrom and Stokey (1982), Geanakoplos (1992) and Morris (1994).[5] Morris (1995), Gul (1998) and Aumann (1998) discuss the common prior assumption. Morris (1996) analyzes the behavior of speculative investors that have heterogeneous priors. Di Tillio (2001) and Morris (2002b) extend Samet's (1998b) technique based on Markovian analysis, while Feinberg (2000) carries out the syntactic characterization of common priors. Lipman (2003) investigates the finite order implications of the common prior assumption. All of these papers present a different approach from ours. They contain neither our diagrammatic representation of cycles, nor the corresponding cycle equations.

Harsanyi (1967-1968) studies games with incomplete information. Mertens and Zamir (1985), and Brandenburger and Dekel (1993) formalize Harsanyi's work by describing universal type spaces. Aumann (1976) introduces the partitional approach

[4] This is the case of what we call *monotonic models* of 2 players when their join partition has only singletons, which is a commonly used class of examples. See our first application. Alternatively, in some games, many cycles may exist. Then, the existence of a common prior will depend on several restrictions involving players' posteriors.

[5] See more references of no trade theorems in Morris (1994).

to study knowledge and common knowledge. He shows that if players have a common prior they cannot agree to disagree, that is, it cannot be common knowledge that they have different posterior probabilities over a given event. Monderer and Samet (1989) generalize the notion of common knowledge, introducing common p-beliefs. The next section describes partitions and beliefs. Section 3 introduces the cycles and their associated equations. Section 4 completes our characterization of posteriors admitting a common prior, while section 5 brings the applications. The appendix contains all proofs.

2 Knowledge and Beliefs

2.1 Partitions

There is a finite set of players, denoted J. Consider **knowledge structures** represented by partitions. Formally, player j's knowledge is given by her **partition** $\Pi^j = \left\{\pi_1^j, \pi_2^j, \cdots, \pi_{l(\Pi^j)}^j\right\}$ of the **state space** S, which is assumed to be a finite set $S = \{1, 2, \cdots, n\}$, with $n \geq 2$, containing all possible worlds (or states of the world). Each subset of the state space belonging to the partition Π^j is called an **atom** of Π^j. Therefore, partition Π^j is a decomposition of S, that is, $S = \pi_1^j \cup \pi_2^j \cup \cdots \cup \pi_{l(\Pi^j)}^j$, with $\pi_i^j \neq \varnothing$, $\forall\ i \in \{1, 2, \cdots, l(\Pi^j)\}$, and $\pi_{i_1}^j \cap \pi_{i_2}^j = \varnothing$, $\forall\, i_1 \neq i_2$, for some positive integer $l(\Pi^j)$. The traditional interpretation is that when the true state of the world is ω, player j learns that the true state lies inside the unique atom of her partition containing ω, which is denoted $\pi^j(\omega)$.[6]

As usual, it is said that partition Π^1 **refines** (or is a refinement of) partition Π^2 if and only if for any $\omega \in S$, the atom containing ω in Π^1 is a subset of the atom containing ω in Π^2, i.e. $\pi^1(\omega) \subset \pi^2(\omega)$. For any collection of partitions $\{\Pi^j \mid\ j \in J\}$, their **join** is defined as the coarsest common refinement of this collection, and it is denoted by $\gamma = \underset{j \in J}{\vee} \Pi^j = \left\{\gamma_1, \cdots, \gamma_{l(\gamma)}\right\}$.[7] Similarly, the **meet** $\delta = \underset{j \in J}{\wedge} \Pi^j = \left\{\delta_1, \cdots, \delta_{l(\delta)}\right\}$ is defined as the finest common coarsening of the collection $\{\Pi^j \mid\ j \in J\}$.

2.2 Posteriors and Priors

Consider the model $(S, \{\Pi^j \mid\ j \in J\})$. The **posterior** of player j is the collection $\theta^j = (\theta_1^j, \cdots, \theta_n^j)$, where, for each state $\omega \in S$ and player $j \in J$, θ_ω^j denotes player j's belief that the true state is ω given that the true state belongs to the atom $\pi^j(\omega)$.[8]

[6] In examples partitions will be represented by Greek letters α, β, ϕ, etc... For more details on knowledge partitions, see Aumann (1976), Geanakoplos (1992a) and (1992b), or Rubinstein (1998).

[7] That is, γ is the only partition of S satisfying both: (i) γ refines Π^j, $\forall\, j \in J$, and (ii) if β refines Π^j, $\forall\ j \in J$, then β refines γ.

[8] Here, the word "posterior" (of player j) represents a collection of probability measures, one measure for each atom of j's partition. By contrast, some authors use this word to express a single

Consider a probability measure $\mu^j = \left(\mu_1^j, \cdots, \mu_n^j\right)$ over the space S and let $\mu^j(B) = \sum_{\omega \in B} \mu_\omega^j$, for any event $B \subset S$. Measure μ^j is a **prior** for player j if for any state $\omega \in S$, whenever $\mu^j\left(\pi^j(\omega)\right) > 0$, then:

$$\theta_\omega^j = \frac{\mu_\omega^j}{\mu^j\left(\pi^j(\omega)\right)} \tag{1}$$

The main question to be studied here is: given posteriors θ^j, with $j \in J$, when is it possible to find a common prior? To simplify the discussion we assume that posteriors have full support.[9]

3 Cycles

The reason why there may exist no common prior is related to the existence of cycles in the meet-join diagram. In this section we describe such diagrams, the cycles and their associated equations.

3.1 Meet-Join Diagram

The next example introduces a diagrammatic representation of the model, called **meet-join diagram**, which helps us to calculate the join and meet of partitions. To help in the visualization, our example has only 2 players. Observe, however, that our technique works in games with any finite number of players. The formal definitions are presented in the next subsection.

Example 1 *(Meet-Join Diagram)*

For any finite space S, because the property

$$(\alpha \vee \beta)(\omega) = \alpha(\omega) \cap \beta(\omega), \qquad \forall\, \omega \in S,$$

holds for every pair of partitions α, β of S, the join and the meet of α and β can be computed with a simple diagram. For example, consider the state space $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and two players with partitions:

$$\alpha = \{\{1, 6\}, \{4, 5\}, \{2, 7, 8\}, \{3\}\}, \qquad \beta = \{\{1, 2\}, \{3, 4\}, \{6, 7\}, \{5\}, \{8\}\}$$

Partitions α and β have 4 and 5 atoms respectively. Both the join $\alpha \vee \beta$, and the meet $\alpha \wedge \beta$ can be calculated using the 4 by 5 matrix representation in figures 1 and 2.

probability measure over the entire space S. In the latter case, such a measure is a function of the true state of the world.

[9] If $\mu^j\left(\pi_r^j\right) = 0$, for some $j \in J$ and $\pi_r^j \in \Pi^j$, it is not possible to use the restriction (1). Then, any posterior for player j is rationalizable inside this atom of her partition. On the other hand, if we assume that $\mu^j\left(\pi_r^j\right) > 0$, for all $j \in J$ and all $\pi_r^j \in \Pi^j$, then, we cannot have $\theta_\omega^{j_1} = 0$ and $\theta_\omega^{j_2} > 0$, for some $\omega \in S$, j_1 and $j_2 \in J$, otherwise there will be no common prior. The only non-trivial case is when players' supports coincide. See Di Tillio (2001) for more on the general case.

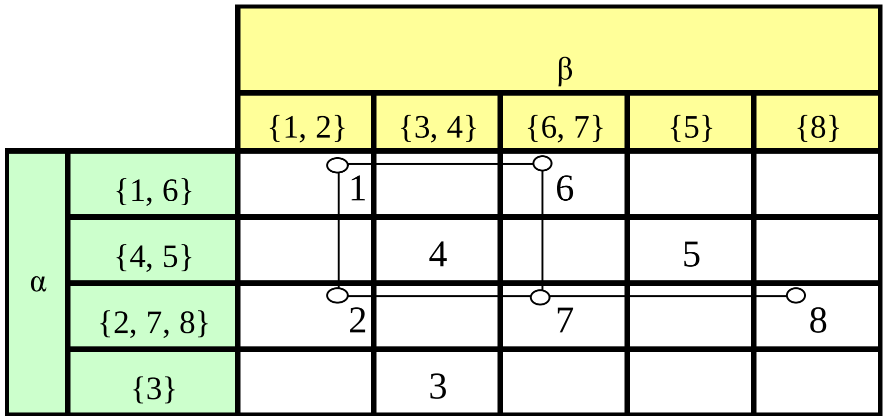

Figure 1: Join and meet, "first" representation.

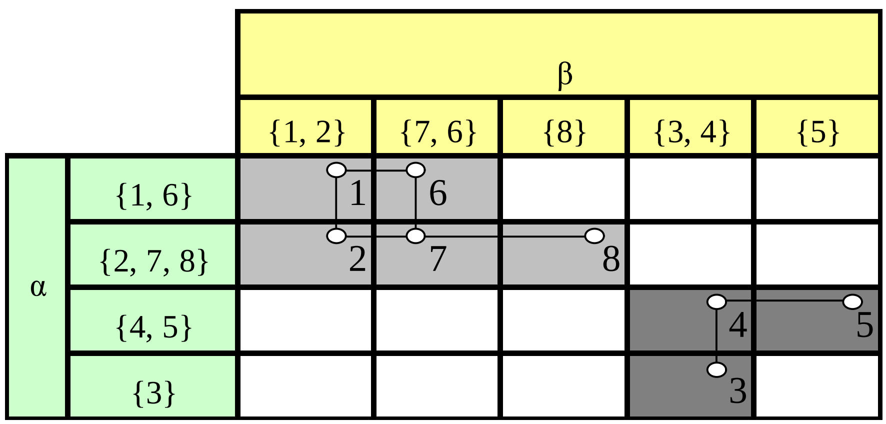

Figure 2: Join and meet, "nice" representation.

Figure 1 *shows partition* α *at the left hand side of a* 4×5 *matrix, and partition* β *on the top of this matrix. Inside the matrix, each state must be written in the same row that contains this state in partition* α*, and in the same column that it appears in partition* β*.*

From this first representation we can calculate the ***connected component*** *of each state. Start at any state, for instance* $\omega = 1$*, and draw all vertical and horizontal lines from state* $\omega = 1$ *to other non-empty cells. From each state that is reached, again, draw all vertical and horizontal lines to other non-empty cells, and so on, until there is no new non-empty cell that can be reached. Every state inside a cell that is reachable from* $\omega = 1$ *is, by definition, in the same connected component that the state* $\omega = 1$ *is. In figure* 1*, states* $\omega = 2$ *and* $\omega = 6$ *are directly reachable and states* $\omega = 7$*, and* $\omega = 8$ *are indirectly reachable from* $\omega = 1$*. Therefore, the connected component of state* $\omega = 1$ *is* $\{1, 2, 6, 7, 8\}$*.*

After finding all connected components, we may redraw the diagram, shifting rows and columns, if necessary, to have each connected component isolated. The result, called "nice" representation, can be seen in figure 2*. The light shaded states (top-left) are in the connected component* $\{1,2,6,7,8\}$*, and the dark shaded states (bottom-right) form the connected component* $\{3,4,5\}$*. The meet* $\alpha \wedge \beta$ *is given by the set of all connected components, that is:*

$$\alpha \wedge \beta = \{\{1,2,6,7,8\}, \{3,4,5\}\}$$

The join $\gamma = \alpha \vee \beta$ *is given by the partition obtained looking at each non-empty cell individually, that is:*

$$\gamma = \{\{1\}, \{2\}, \{6\}, \{7\}, \{8\}, \{4\}, \{3\}, \{5\}\}$$

There are several ***edges*** *and* ***paths*** *in the diagram.*[10] *For instance, the path* c *below may be represented by the following sequence of edges:*

$$c = ([1,6], [6,7], [7,2], [2,1])$$

It turns out that path c *is a cycle. On the other hand, the path* $([8,7], [7,6], [6,7], [7,8])$ *will not be considered to be a cycle because the state* $\omega = 7$ *is present in more than* 2 *different edges.* ■

3.2 Edges, Paths and Cycles

We introduce some necessary notation now.

Definition 1 *(Edges, Paths and Cycles)*

Consider the model $(S, \{\Pi^j \mid j \in J\})$ *and recall that* γ *represents the join partition, i.e.* $\gamma = \Pi^1 \vee \cdots \vee \Pi^J = \{\gamma_1, \cdots, \gamma_{l(\gamma)}\}$.

(a) An (oriented) ***edge*** $x_{p,q} = [\omega_p, \omega_q]$*, with* ω_p*,* $\omega_q \in S$*,* $p \neq q$*, is an ordered pair of states, such that* ω_p *and* ω_q *are in the same atom of partition* Π^j *for some player* $j \in J$*. In this case, we say that* $x_{p,q}$ *has* ***direction*** j*.*[11]

(b) The edge x_{p_2,q_2} *is said to be* ***consecutive*** *to the edge* x_{p_1,q_1} *if and only if* $q_1 = p_2$*. Define the* ***opposite*** *of edge* $x_{p,q}$*, denoted* $-x_{p,q}$*, by* $-x_{p,q} = x_{q,p}$*.*

(c) A ***path*** *is an* k*-tuple* $x = (x_{0,1}, x_{1,2}, \cdots, x_{k-1,k})$ *of consecutive edges such that for all* $i \in \{1, \cdots, k-1\}$*,* $x_{i-1,i}$ *and* $x_{i,i+1}$ *have at least one different direction. Let* $-x$ *denote the opposite of path* x*, that is:*

$$-x = (-x_{k-1,k}, -x_{k-2,k-1}, \cdots, -x_{1,2}, -x_{0,1})$$

(d) The states ω_1 *and* $\omega_2 \in S$ *are said to be in the same* ***connected component*** *of the meet-join diagram if and only if there is a path from* ω_1 *to* ω_2*.*

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[10] See definition 1 ahead.

[11] An edge may have many directions.

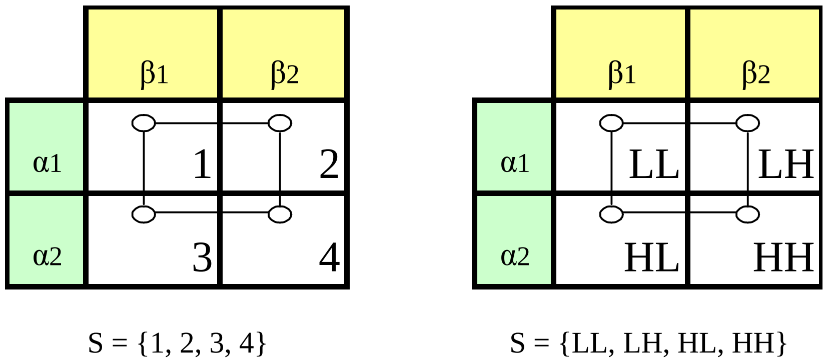

Figure 3: Two versions of the meet-join diagram in the duopoly example.

*(e) A path $c = (x_{0,1}, x_{1,2}, \cdots, x_{k-2,k-1}, x_{k-1,k})$ is said to be a **cycle** if and only if it has at least 2 edges, the edge $x_{0,1}$ is consecutive to $x_{k-1,k}$ and each state in S belongs to at most two edges of c.*[12] *By definition, we will regard any "rotation of c", such as $(x_{1,2}, \cdots, x_{k-2,k-1}, x_{k-1,k}, x_{0,1})$ or $(x_{k-1,k}, x_{0,1}, x_{1,2}, \cdots, x_{k-2,k-1})$, as the same cycle as c.*[13]

3.3 Cycle Equations

Definition 2 *(Cycle Equations)*

Consider the model $\left(S, \left\{(\Pi^j, \theta^j) \mid \ j \in J\right\}\right)$. For each cycle, if one exists, we will associate an equation to it. Consider the cycle c, with $k \geq 2$ edges, given by:

$$c = (x_{1,2}, x_{2,3}, x_{3,4}, \cdots, x_{k-1,k}, x_{k,1}),$$

where $x_{i,i+1} = [\omega_i, \omega_{i+1}]$, and $x_{i,i+1}$ has direction j_i, with $i \in \{1, \cdots, k\}$.[14] *Then, the associated **cycle equation** is defined as:*

$$\theta^{j_k}_{\omega_1}.\theta^{j_1}_{\omega_2}.\theta^{j_2}_{\omega_3} \cdots \theta^{j_{k-1}}_{\omega_k} = \theta^{j_1}_{\omega_1}.\theta^{j_2}_{\omega_2}.\theta^{j_3}_{\omega_3} \cdots \theta^{j_k}_{\omega_k}$$

The next example provides some intuition for cycle equations.

Example 2 *(Duopoly, Part 1*[15]*)*

Imagine that each duopolist in a Cournot game knows her own marginal cost, which may be either low (L) or high (H), but is uncertain about her opponent's cost.

[12] This last requirement does not allow that in a cycle we move from one state, say ω_1, to another, say ω_2, and then to a third state, and then return to ω_2 and ω_1. See the last part of example 1.

[13] There is some language abuse in the definition of a cycle. Formally, a cycle would be an equivalence class of paths, under the "rotation" relation.

[14] Because we define state ω_{k+1} as $\omega_{k+1} = \omega_1$, then $x_{k,k+1} = x_{k,1}$.

[15] Geanakoplos and Polemarchakis (1982), Gul (1998), Morris (2002b), among others, used the partitions in this example.

If a player has low (respectively high) cost, we refer to this player as a low (high) type. To model this game, let $S = \{1, 2, 3, 4\}$, and players' partitions be denoted by α and β respectively, where $\alpha = \{\alpha_1, \alpha_2\}$, $\alpha_1 = \{1, 2\}$, $\alpha_2 = \{3, 4\}$, $\beta = \{\beta_1, \beta_2\}$, $\beta_1 = \{1, 3\}$, and $\beta_2 = \{2, 4\}$. We define types for the players as follows: in states $\omega \in \{1, 2\}$ player 1 is low type, and in states $\omega \in \{3, 4\}$ she is high type. Player 2 has low type if $\omega \in \{1, 3\}$, and high type if $\omega \in \{2, 4\}$. States may be relabeled as:

$$LL = 1, \qquad LH = 2, \qquad HL = 3, \qquad \text{and} \qquad HH = 4$$

Figure 3 shows two versions of the meet-join diagram associated with this example. The version on the right hand side uses the new labels for the states, namely LL, LH, HL, HH.

For a common prior to exist, posteriors must satisfy the following restriction:[16]

$$\theta^2_3.\theta^1_4.\theta^2_2.\theta^1_1 = \theta^1_2.\theta^2_4.\theta^1_3.\theta^2_1 \tag{2}$$

The expression at the left hand side of (2) may be interpreted as running over the cycle of diagram 3 in counterclockwise direction, starting at the state $\omega = 1$ (top-left), and going down, right, up and left. The right hand side may be viewed as starting from state $\omega = 1$, and moving in the clockwise direction (right, down, left, up). The cycle equation (2) is requiring that running over the cycle in both directions results in the same value.

Another way of describing the left hand side of the cycle equation (2) is the posterior that player 2 has, at her π^2_1 atom, that player 1 is in her π^1_2 atom, i.e. θ^2_3, thinking that 2 is in her π^2_2 atom, i.e. θ^1_4, thinking that player 1 is in her π^1_1 atom, i.e. θ^2_2, thinking that 2 is in her π^2_1 atom, i.e. θ^1_1. The right hand side has a similar interpretation. ■

The following proposition links cycle equations, and consequently the meet-join diagram, to the problem of common prior existence.

Proposition 1 *(Cycle Equation Necessity)*
Consider the model $\left(S, \left\{(\Pi^j, \theta^j) \mid \ j \in J\right\}\right)$, where S is finite. Then, posteriors may be rationalized by a common prior only if all cycle equations are satisfied.

4 Common Prior Existence

So far, we have presented necessary conditions on the posteriors for making them rationalizable by a common prior. Now, we show that these conditions are sufficient as well by studying a certain system of linear equations.

[16] See propositions 1 and 2. The interpretation of this condition is presented below in this example and in applications 1 and 2 ahead.

4.1 The System

For each player $j \in J$ and index $r \in \{1, \cdots, l(\Pi^j)\}$, let λ_r^j denote j's prior that the true state is inside the atom π_r^j, that is:

$$\lambda_r^j = \mu^j(\pi_r^j) \tag{3}$$

Let $\lambda^j = (\lambda_1^j, \cdots, \lambda_{l(\Pi^j)}^j)$, $\forall\, j \in J$. Each λ_r^j, with $j \in J$ and $r \in \{1, \cdots, l(\Pi^j)\}$, will represent a typical variable in the system that we describe next. Let $r_j(\omega)$ be the index of the unique atom in j's partition such that $\omega \in \pi_{r_j(\omega)}^j$. Thus, it is possible to find a common prior if and only if the following system of linear equations

$$\lambda_{r_1(\omega)}^1 \theta_\omega^1 = \lambda_{r_2(\omega)}^2 \theta_\omega^2 = \cdots = \lambda_{r_j(\omega)}^j \theta_\omega^j = \cdots = \lambda_{r_J(\omega)}^J \theta_\omega^J, \qquad \forall\, \omega \in S, \tag{4}$$

$$\sum_{r=1}^{l(\Pi^1)} \lambda_r^1 = 1 \tag{5}$$

has at least one solution $\lambda^j \in \mathbb{R}_+^{l(\Pi^1)} \times \cdots \times \mathbb{R}_+^{l(\Pi^J)}$.[17]

Definition 3 *(System of Candidate Equations)*
The linear system composed by equations (4) and (5) will be called the ***system of candidate equations****.*

When there is a common prior, say $\widetilde{\mu}$, the system of candidate equations has a solution. In particular, (4) holds. Hence, for any player $j \in J$, the value $\lambda_{r_j(\omega)}^j \theta_\omega^j$ will be equal to the common prior that the true state is ω. Mathematically:

$$\widetilde{\mu_\omega} = \lambda_{r_1(\omega)}^1 . \theta_\omega^1 = \cdots = \lambda_{r_j(\omega)}^j . \theta_\omega^j = \cdots = \lambda_{r_J(\omega)}^J . \theta_\omega^J, \qquad \forall\ \omega \in S \tag{6}$$

Example 3 *(Duopoly, Part 2)*

[17] Our system will have exactly $n(J-1)+1$ equations. All equations of the form $\sum_{r=1}^{l(\Pi^j)} \lambda_r^j = 1$, for $j \in J - \{1\}$, may be mathematically deduced from our system. To see why, consider the case of a two-player game. Adding all equations in (4), we obtain:

$$\sum_{r=1}^{l(\Pi^1)} \lambda_r^1 = \sum_{r=1}^{l(\Pi^1)} \sum_{k \in \pi_r^1} \lambda_r^1 \theta_k^1 = \sum_{s=1}^{l(\Pi^2)} \sum_{k \in \pi_s^2} \lambda_s^2 \theta_k^2 = \sum_{s=1}^{l(\Pi^2)} \lambda_s^2$$

Because the summation in (5) is 1, the analogous summation for player 2 must also be 1. Here, we are using the fact that $\sum_{\omega \in \pi^j(k)} \theta_\omega^j = 1$, for any player j and any state $k \in S$.

Again, consider the duopoly model $(S, \{\alpha, \beta\})$, *where* $\alpha = \{\alpha_1, \alpha_2\}$, $\alpha_1 = \{1, 2\}$, $\alpha_2 = \{3, 4\}$, $\beta = \{\beta_1, \beta_2\}$, $\beta_1 = \{1, 3\}$, *and* $\beta_2 = \{2, 4\}$. *Applying (4) and (5), the system of candidate equations becomes:*

$$\left\{\begin{array}{c} \theta_1^1.\lambda_1^1 = \theta_1^2.\lambda_1^2 \\ \theta_2^1.\lambda_1^1 = \theta_2^2.\lambda_2^2 \\ \theta_3^1.\lambda_2^1 = \theta_3^2.\lambda_1^2 \\ \theta_4^1.\lambda_2^1 = \theta_4^2.\lambda_2^2 \\ \lambda_1^1 + \lambda_2^1 = 1 \end{array}\right. \tag{7}$$

There are 4 *variables, namely* λ_1^1, λ_2^1, λ_1^2, *and* λ_2^2. *To obtain a solution, posteriors must satisfy the cycle equation (2). If (2) holds, the system has exactly* 4 *linearly independent equations, and a unique solution. In this case, it is possible to obtain a common prior. If (2) is not satisfied, the system (7) has* 5 *linearly independent equations and there is no common prior.*[18] ■

4.2 Solution Existence

The existence of cycles creates the possibility that too many candidate equations are linearly independent. If there are more linearly independent candidate equations than variables, the system will not have a solution. However, when all cycle equations are satisfied, there will be at most as many independent candidate equations as the number of variables. In this case, a common prior exists. Proposition 2 below is our main result.

Proposition 2 *(Cycle Equations: Sufficiency)*
Consider the model $\left(S, \left\{(\Pi^j, \theta^j) \mid \ j \in J\right\}\right)$, *where* S *is finite. If all cycle equations are satisfied, then posteriors may be rationalized by a common prior.*

The argument for proving the sufficiency of cycle equations is in the appendix. Its basic idea is presented in the following example.

Example 4 *(Duopoly, Part 3)*

[18] If the first 4 equations were independent, then $\lambda_1^1 = 0 = \lambda_2^1$. But this would violate the equation $\lambda_1^1 + \lambda_2^1 = 1$. Morris (2002b) used the following particular values for the posteriors:

$$\theta_1^1 = 1, \quad \theta_2^1 = 0, \quad \theta_3^1 = 0, \quad \theta_4^1 = 1, \quad \text{and} \quad \theta_1^2 = 0, \quad \theta_2^2 = 1, \quad \theta_3^2 = 1, \quad \theta_4^2 = 0.$$

He observed that it is not possible to have a common prior because there is no convergence of iterated expectations. Using these particular values in the cycle equation (2):

$$\theta_1^1.\theta_2^2.\theta_3^2.\theta_4^1 = 1 \neq 0 = \theta_1^2.\theta_2^1.\theta_3^1.\theta_4^2$$

Consequently, by proposition 1, for these particular posteriors, it is not possible to find a common prior.

Again, consider the duopoly model. Assume that the cycle equation (2) holds. Consider the function $h : [0, +\infty) \to \mathbb{R}$, defined as:

$$h(\overline{\lambda_1^1}) = 1 - \overline{\lambda_2^1},$$

where $\overline{\lambda_1^1} \in [0, +\infty)$ and the value $\overline{\lambda_2^1}$ is a function of $\overline{\lambda_1^1}$ given by either one of the formulas:[19]

$$\overline{\lambda_2^1} = \left(\frac{\theta_2^1.\theta_4^2}{\theta_4^1.\theta_2^2}\right)\overline{\lambda_1^1}, \qquad \text{or} \qquad \overline{\lambda_2^1} = \left(\frac{\theta_1^1.\theta_3^2}{\theta_3^1.\theta_1^2}\right)\overline{\lambda_1^1}$$

Because the cycle equation (2) holds, the value λ_2^1 is the same regardless of which formula above we use to calculate it. Hence, the function $h(\cdot)$ is well defined. It is also decreasing, with $h(0) = 1$ and $\lim\limits_{\overline{\lambda_1^1} \to +\infty} h(\overline{\lambda_1^1}) = -\infty$. We conclude that $h(\cdot)$ has a unique fixed point. This fixed point, denoted λ_1^1, together with the corresponding λ_2^1 form part of the solution to the system (7). The values of λ_1^2 and λ_2^2 can be calculated by (7). Thus, the common prior $\widetilde{\mu}$ may be calculated by (6).

4.3 Uniqueness

If it exists, is the common prior unique? It depends on how many solutions the system of candidate equations has, or equivalently, on how many connected components there are.

Assume that all cycle equations hold. If the meet-join diagram has a single connected component, then, there is a unique solution to the system of candidate equations, and therefore, a unique common prior. When there is more than a single connected component (i.e. the meet has more than a single atom), there are infinitely many solutions to the system. However, solutions are unique up to the assignment of arbitrary weights $\rho_m = \mu^j(\delta_m) > 0$, with $m \in \{1, \cdots, l(\delta)\}$ and $\sum\limits_{m=1}^{l(\delta)} \rho_m = 1$, to the connected components.[20] Because $\sum\limits_{m=1}^{l(\delta)} \rho_m = 1$, any set of $l(\delta) - 1$ weights uniquely determines the remaining weight. Thus, the dimension of the space of solutions equals to $l(\delta) - 1$.

5 Applications

In this section, we present a few examples. To simplify matters, part of the material here is presented for games of 2 players only. Observe, however, that this is to facilitate

[19] To find the two formulas for $\overline{\lambda_2^1}$ as a function of $\overline{\lambda_1^1}$, two equations of the system (7) were used in the first formula, and two other equations were used in the second formula.

[20] Each posterior θ_ω^j acts inside a unique atom of the meet. Hence, posteriors have no saying in the assignment of the weights ρ_m. See an example in the appendix.

the exposition and we can generalize it to games with more than 2 players.

5.1 Monotonic Models

If the state space is endowed with a linear order such that the partitions of all players have only consecutive states in each atom, the model is called monotonic. In this case, the respective meet-join diagram for games of 2 players is "stairs" shaped (shifting rows and columns, if necessary), and cycles may only occur inside atoms of the join. In the particular case when the join has only singletons, there will be no cycles, and then, any posteriors may be rationalized by a common prior. This class of examples contains all games of 2 players in which one player has a knowledge partition that refines the other player's partition.[21] In particular, it includes all 2-player games with one-sided uncertainty. The e-mail game is another example of monotonic model whose join partition is made of singletons only.[22] ■

5.2 Optimism in the Duopoly Game

We return to the Cournot duopoly game. Recall that each one of two duopolists knows her own marginal cost, which may be either low or high, but is uncertain about her opponent's cost. A player is low (respectively high) type if and only if she has low (respectively high) marginal cost, denoted c_L (respectively c_H). More precisely, let $0 \leq c_L < c_H$, with c_H being sufficiently small so that both players have positive profits in all states. As we show in the appendix, when a player observes her own type, being a low type is good news because her profits are higher, regardless of her opponent's type.[23]

From player 1's viewpoint, the ratio of the probabilities that her opponent is high type, with respect to being a low type, is $\frac{\theta_2^1}{\theta_1^1}$ when player 1 herself is low type, and $\frac{\theta_4^1}{\theta_3^1}$ when she is high type. The greater $\frac{\theta_2^1}{\theta_1^1}$ is, the larger player 1's expected profits will be when she is low type. Similarly, the greater $\frac{\theta_4^1}{\theta_3^1}$ is, the larger player 1's expected profits when she is high type.

Definition 4 *(Optimism)*

*The **optimism** of player $j \in J$ in the duopoly game, denoted Opt^j, is defined as:*

$$Opt^1 = \frac{\theta_2^1/\theta_1^1}{\theta_4^1/\theta_3^1} \qquad \textit{and} \qquad Opt^2 = \frac{\theta_3^2/\theta_1^2}{\theta_4^2/\theta_2^2}$$

[21] For instance, see Levin (2001).

[22] The e-mail game is introduced in Rubinstein (1989). See also Samuelson and Binmore (2001), and Morris (2002a). Observe, however, that the e-mail game has infinite states, so our result does not apply.

[23] Also good news would be the fact that her opponent is high type, but this information is not available. See the formal analysis of the duopoly game in the appendix.

Opt^1 measures how much the ratio of player 1's posterior that her opponent is a high type, over the posterior that she is a low type, increases if player 1's own type would change from high to low.

Now, we have a general interpretation for the cycle equation (2). It may be rewritten as

$$Opt^1 = Opt^2$$

In words, the cycle equation is saying that the optimism of both players must coincide.[24] This shows that the possibility of common prior existence is associated with a behavioral assumption over the players. This is evidence that the possibility of common prior existence is not only a naive technical modeling assumption. It may also impose restrictions on behavior.

Observe that a sufficient condition for the cycle equation (2) to hold is that, for each player, the posterior that her opponent is a high type, over the posterior that she is a low type, is independent of her own type. Mathematically:

$$\frac{\theta_2^1}{\theta_1^1} = \frac{\theta_4^1}{\theta_3^1} \qquad \text{and} \qquad \frac{\theta_3^2}{\theta_1^2} = \frac{\theta_4^2}{\theta_2^2}$$

5.3 Beliefs that are Type Independent

We generalize the last observation about beliefs that are independent of the players' own types.[25] Formally, for each player $j \in J$, let $t^j \in T^j$ denote j's type, where $T^j = \{1, 2, \cdots, m^j\}$, for some integer $m^j \geq 2$, is the set of j's possible types. Knowledge partitions are such that each player knows her own type, but do not know her opponents' types. Hence, the state space can be written as $S = \bigotimes_{j \in J} T^j$. Let t^{-j} denote the profile of players's type, except for player j.

Definition 5 *(Independent Beliefs)*

We say that player j has ***independent beliefs*** *if for every profile $t^{-j} \in \bigotimes_{i \in J - \{j\}} T^i$, then j's posterior about other players' types is independent of her own type.*

The next proposition explains the relation between independent beliefs and the common prior existence.

---

[24] Because our conditions require exact values for posteriors, a generic perturbation destroys our cycle equation, $Opt^1 = Opt^2$. In this sense, the existence of a common prior seems to be very special. Note, however, that among the collection of all binary relations that a set has, preference relations are also a very special case. Nevertheless, most economic theory is built assuming that agents have preference relations. See Morris (1995) for a deeper discussion of the common prior assumption in economics.

[25] Here, the type of a player represents the states of the world inside a given atom of this player's knowledge partition. Some authors reserve the word type to describe the full hierarchy of beliefs that the player has in the universal type space. See Mertens and Zamir (1985), or Brandenburger and Dekel (1993).

Proposition 3 *(Independent Beliefs Imply Common Prior[26])*

If all players have independent beliefs, then all cycle equations hold, and therefore, posteriors may be rationalized by a common prior.

5.4 Bets

Samet (1998a) presents a version of a famous agreement result. He shows that a necessary and sufficient condition for the **non**-existence of bets, with common knowledge that all players have positive expected gains, is that posteriors may be rationalized by a common prior. But we have presented necessary and sufficient conditions for this to be true, namely the cycle equations. Therefore, we immediately have the following corollary.

Corollary 1 *(Bets)*

All cycle equations are satisfied if and only if there is no bet for which it is always common knowledge that all players have positive expected gains.

This corollary provides an easy to check sufficient condition for the non-existence of bets in which it is common knowledge that players have positive expected gain, in terms of the partitions. If players' partitions have no cycles, we automatically see that there will be no bets. This is the case of monotonic models of 2-player games when the join has only singletons, for instance.

5.4.1 Bets over a Cycle - General Framework

Consider a model $\left(S, \left\{(\Pi^j, \theta^j) \mid \ j \in J\right\}\right)$ and the cycle c, with $k \geq 2$ edges, given by:

$$c = (x_{1,2}, x_{2,3}, x_{3,4}, \cdots, x_{k-1,k}, x_{k,1}),$$

where the edge $x_{i,i+1} = [\omega_i, \omega_{i+1}]$ has direction j_i, with $i \in \{1, \cdots, k\}$.[27] Hence, states ω_i and ω_{i+1} are in the same atom of partition Π^{j_i}, for any player j_i in the cycle c. The associated cycle equation is:

$$\theta^{j_k}_{\omega_1}.\theta^{j_1}_{\omega_2}.\theta^{j_2}_{\omega_3} \cdots \theta^{j_{k-1}}_{\omega_k} = \theta^{j_1}_{\omega_1}.\theta^{j_2}_{\omega_2}.\theta^{j_3}_{\omega_3} \cdots \theta^{j_k}_{\omega_k}$$

Let J_c represent the ordered sequence of players that compose the cycle c, that is, $J_c = (j_1, j_2, \cdots, j_k)$, with all $j_i \neq j_{i+1}$ and $j_1 \neq j_k$.

Let $g^{j_i}_{\omega}$ represent the payoff of player j_i, at state ω, in a bet in which j_i wins in state ω_{j_i}, loses in state $\omega_{j_{i+1}}$, and makes zero profit in all other states. In other words, at ω_{j_i}, player j_{i+1} transfers $g^{j_i}_{\omega}$ to player j_i. Let the expected payoff of each player

[26] Thanks to Claudio Mezzetti who conjectured the general result.

[27] We define state ω_{k+1} as $\omega_{k+1} = \omega_1$. Hence, $x_{k,k+1} = x_{k,1}$. In fact, the players in the cycle could appear more than once, as long as each player is not present in consecutive positions of the cycle.

$j_i \in J_c$, at the atom of her partition that contains the states ω_{j_i} and $\omega_{j_{i+1}}$, be denoted E^{j_i}. Then:

$$E^{j_1} = \theta^{j_1}_{\omega_1}.g^{j_1}_{\omega_1} + \theta^{j_1}_{\omega_2}.g^{j_1}_{\omega_2} \tag{8a}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$E^{j_i} = \theta^{j_i}_{\omega_i}.g^{j_i}_{\omega_i} + \theta^{j_i}_{\omega_{i+1}}.g^{j_i}_{\omega_{i+1}} \tag{8b}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$E^{j_k} = \theta^{j_k}_{\omega_k}.g^{j_k}_{\omega_k} + \theta^{j_k}_{\omega_1}.g^{j_k}_{\omega_1} \tag{8c}$$

Transfers $g^{j_i}_{\omega}$ are such that:

$$g^{j_1}_{\omega_2} = -g^{j_2}_{\omega_2} \tag{9a}$$

$$\cdots\cdots\cdots\cdots$$

$$g^{j_i}_{\omega_{i+1}} = -g^{j_{i+1}}_{\omega_{i+1}} \tag{9b}$$

$$\cdots\cdots\cdots\cdots$$

$$g^{j_k}_{\omega_1} = -g^{j_1}_{\omega_1} \tag{9c}$$

We will normalize the payoffs in such a way that $g^{j_1}_{\omega_1} = 1$. Using this normalization and solving the system of equations (8) and (9), we obtain that for each $j \in J$:

$$g^{j_i}_{\omega_{i+1}} = \frac{E^{j_i}}{\theta^{j_i}_{\omega_{i+1}}} + \frac{\theta^{j_i}_{\omega_i}.E^{j_{i-1}}}{\theta^{j_{i-1}}_{\omega_i}.\theta^{j_i}_{\omega_{i+1}}} + \frac{\theta^{j_{i-1}}_{\omega_{i-1}}.\theta^{j_i}_{\omega_i}.E^{j_{i-2}}}{\theta^{j_{i-2}}_{\omega_{i-1}}.\theta^{j_{i-1}}_{\omega_i}.\theta^{j_i}_{\omega_{i+1}}} + \cdots + \frac{\theta^{j_2}_{\omega_2}\cdots\theta^{j_i}_{\omega_i}.E^{j_1}}{\theta^{j_1}_{\omega_2}\cdots\theta^{j_i}_{\omega_{i+1}}} - \frac{\theta^{j_1}_{\omega_1}.\theta^{j_2}_{\omega_2}\cdots\theta^{j_i}_{\omega_i}}{\theta^{j_1}_{\omega_2}\cdots\theta^{j_i}_{\omega_{i+1}}}$$

$$g^{j_i}_{j_i} = \frac{-E^{j_i}}{\theta^{j_i}_{\omega_{i+1}}} - \frac{\theta^{j_i}_{\omega_i}.E^{j_{i-1}}}{\theta^{j_{i-1}}_{\omega_i}.\theta^{j_i}_{\omega_{i+1}}} - \frac{\theta^{j_{i-1}}_{\omega_{i-1}}.\theta^{j_i}_{\omega_i}.E^{j_{i-2}}}{\theta^{j_{i-2}}_{\omega_{i-1}}.\theta^{j_{i-1}}_{\omega_i}.\theta^{j_i}_{\omega_{i+1}}} - \cdots - \frac{\theta^{j_2}_{\omega_2}\cdots\theta^{j_i}_{\omega_i}.E^{j_1}}{\theta^{j_1}_{\omega_2}\cdots\theta^{j_i}_{\omega_{i+1}}} + \frac{\theta^{j_1}_{\omega_1}.\theta^{j_2}_{\omega_2}\cdots\theta^{j_i}_{\omega_i}}{\theta^{j_1}_{\omega_2}\cdots\theta^{j_i}_{\omega_{i+1}}}$$

Therefore, the generalized expectations hiper-plane becomes:

$$\frac{E^{j_1}}{\eta^1} + \frac{E^{j_2}}{\eta^2} + \frac{E^{j_3}}{\eta^3} + \cdots + \frac{E^{j_k}}{\eta^k} = \Delta, \tag{10}$$

where:

$$\Delta = \frac{\theta^{j_1}_{\omega_1}.\theta^{j_2}_{\omega_2}\cdots\theta^{j_k}_{\omega_k} - \theta^{j_1}_{\omega_2}.\theta^{j_2}_{\omega_3}\cdots\theta^{j_k}_{\omega_1}}{\theta^{j_1}_{\omega_2}.\theta^{j_2}_{\omega_3}\cdots\theta^{j_k}_{\omega_1}},$$

and

$$\eta^i = \frac{\theta^{j_i}_{\omega_{i+1}}\cdots\theta^{j_k}_{\omega_1}}{\theta^{j_{i+1}}_{\omega_{i+1}}\cdots\theta^{j_k}_{\omega_k}}, \quad \text{with} \quad i \in \{1, \cdots, k\}$$

In particular, if $\Delta > 0$, it is possible to find positive values E^{j_i}, with $j_i \in J_c$, satisfying (10).

5.4.2 Bets in 2-Player Cycles

Again, consider 2 players with the same knowledge partitions of the Cournot duopoly example. Can they form a bet in such a way that one of them wins in a particular state, say ω_0, and loses in all other 3 states? No because the player betting in ω_0 would not bet if she would lose for sure. Hence, by accepting the bet, the player betting in ω_0 would reveal to her opponent in which atom of her partition she is in. Thus, her opponent would accept the bet only if she is sure to win. Anticipating this, the former player would never accept the bet.

A similar reasoning indicates that we can never expect to see a bet where one of the players wins in states $\omega \in \{1, 2\}$ and loses in states $\omega \in \{3, 4\}$ (because player 1 would reveal her atom), or bets in which one of the players wins in states $\omega \in \{1, 3\}$ and loses in states $\omega \in \{2, 4\}$ (because player 2 would reveal her atom).[28] We conclude that the only bet in which it is always common knowledge that all players have positive expected gains involves one of the players winning in states $\omega \in \{1, 4\}$ and losing in states $\omega \in \{2, 3\}$. This means that one player is betting that both players have the same type, either $\omega = LL$ or $\omega = HH$, and the other player is betting that they have different types, either $\omega = LH$ or $\omega = HL$.

Consider a lottery:

$$G = (g_1, g_2, g_3, g_4)\,,$$

where $g_i \in \mathbb{R}$ represents the transfer from player 2 to player 1 if the true state of the world ω reveals itself to be $\omega = i$.[29] Let $E^1G(\alpha_r)$ and $E^2[-G(\beta_s)]$ denote the expected gains of players 1 and 2, respectively, under lottery G, when they are in atoms α_r and β_s of their knowledge partitions. Since the meet $\alpha \wedge \beta$ is equal to S, it is common knowledge that both players have positive expected gains if and only if they have positive expected profits regardless of the true state. Mathematically:

$$E^1\left[G(\alpha_1)\right] = \theta^1_1.g_1 + \theta^1_2.g_2 > 0, \qquad E^1\left[G(\alpha_2)\right] = \theta^1_3.g_3 + \theta^1_4.g_4 > 0,$$

$$E^2[-G(\beta_1)] = \theta^2_1.(-g_1) + \theta^2_3.(-g_3) > 0, \qquad E^2[-G(\beta_2)] = \theta^2_2.(-g_2) + \theta^2_4.(-g_4) > 0$$

Bets are only possible if there is no possible common prior. In a previous section we learned that this means players have different optimism. If $Opt^1 < Opt^2$, player 1 wins the bet whenever the true state belongs to $\{LL, HH\}$, and player 2 wins in $\{LH, HL\}$.[30]

5.4.3 2-Player Cycles with a Market Maker

Unless $Opt^1 = Opt^2$ holds, there is an arbitrage opportunity, that is, there is "space" for a third party, for instance a market maker, to intermediate a bet, even if this ad-

[28] Again, in any such bet one party would only accept to enter if she knows that she will win for sure, so the other player refuses to bet.

[29] A negative transfer means that player 1 is the one who is transferring money to player 2.

[30] Conversely, if $Opt^1 > Opt^2$ player 1 wins the bet if the true state belongs to $\{LH, HL\}$ and player 2 wins in $\{LL, HH\}$.

ditional player, called player 3, has no clue about the true state. Formally, we assume that player 3 is incapable of making any probabilistic estimation of the likelihood of any state of S.

Suppose that $Opt^1 < Opt^2$. For each $\omega \in S$, let g^1_ω be the transfer from player 3 to player 1 if the true state turns out to be ω. Similarly, for each $\omega \in S$, let g^2_ω be the transfer from player 2 to player 3 if the true state turns out to be ω. Thus, the third party's profit at state ω, denoted g^3_ω, is:

$$g^3_\omega = g^2_\omega - g^1_\omega, \qquad \forall\, \omega \in S \tag{11}$$

To simplify the calculation, suppose that player 3 offers bets such that players 1 and 2's expected gains are independent of the atom they are, i.e. $\varepsilon^1 = E^1G(\alpha_1) = E^1G(\alpha_2) > 0$ and $-\varepsilon^2 = E^2[-G(\beta_1)] = E^2[-G(\beta_2)] > 0$. This hypothesis is not restrictive. In this case:

$$\varepsilon^1 = \theta^1_1.g^1_1 + \theta^1_2.g^1_2 = \theta^1_3.g^1_3 + \theta^1_4.g^1_4, \tag{12}$$

$$-\varepsilon^2 = \theta^2_1.(-g^2_1) + \theta^2_3.(-g^2_3) = \theta^2_2.(-g^2_2) + \theta^2_4.(-g^2_4) \tag{13}$$

Changing units if necessary, we can assume that one of the payoffs is unitary. Suppose $g^1_4 = 1$. Then, solving the system of equations (11), (12) and (13), we find that the profit vector $g^3 = (g^3_1, g^3_2, g^3_3, g^3_4)$ of player 3 will be on the hyper-plane:

$$\begin{aligned} &\frac{g^3_1}{\theta^1_2\theta^2_3\theta^2_4} + \frac{g^3_2}{\theta^1_1\theta^2_3\theta^2_4} + \frac{g^3_3}{\theta^1_2\theta^2_1\theta^2_4} + \frac{g^3_4}{\theta^1_1\theta^2_2\theta^2_3} = \\ &= \Delta + \varepsilon^1.\left(\frac{\theta^1_3.\theta^2_1 - \theta^1_1.\theta^2_3}{\theta^1_1.\theta^1_2.\theta^1_3.\theta^2_1.\theta^2_3.\theta^2_4}\right) - \varepsilon^2.\left(\frac{\theta^1_2.\theta^2_1 - \theta^1_1.\theta^2_2}{\theta^1_1.\theta^1_2.\theta^2_1.\theta^2_2.\theta^2_3.\theta^2_4}\right), \end{aligned} \tag{14}$$

where:

$$\Delta = \frac{\theta^1_1.\theta^2_2.\theta^2_3.\theta^1_4 - \theta^2_1.\theta^1_2.\theta^1_3.\theta^2_4}{\theta^1_1.\theta^1_2.\theta^1_3.\theta^2_1.\theta^2_2.\theta^2_3.\theta^2_4} = \frac{(Opt^2 - Opt^1)\,\theta^1_4}{\theta^1_2.\theta^1_3.\theta^2_2.\theta^2_3}$$

Choosing ε^1 and ε^2 sufficiently close to zero, this hyper-plane approaches arbitrarily well the hyper-plane:

$$\frac{g^3_1}{\theta^1_2\theta^2_3\theta^2_4} + \frac{g^3_2}{\theta^1_1\theta^2_3\theta^2_4} + \frac{g^3_3}{\theta^1_2\theta^2_1\theta^2_4} + \frac{g^3_4}{\theta^1_1\theta^2_2\theta^2_3} = \Delta,$$

which intercepts the origin if and only if $\Delta = 0$, or equivalently $Opt^1 = Opt^2$, that is, exactly when it is possible that posteriors may come from a common prior.

Assuming that $Opt^1 < Opt^2$, then $\Delta > 0$. Hence, the market maker can make a risk free profit by choosing any g^3 that satisfies (14), $g^3_\omega > 0$, for all $\omega \in S$, and sufficiently small and positive values for ε^1 and $-\varepsilon^2$.

6 Appendix - Proofs

Proof. (Proposition 1 - Cycle Equations: Necessity)

Consider the cycle $c = ([1,2],[2,3],\cdots,[k,1])$, with $k \geq 2$. For all $i \in \{1,\cdots,k\}$, states i and $i+1$ are in the same atom of partition Π^{j_i}, for a collection of (not necessarily distinct) players $\{j_1,\cdots,j_k\}$, with $j_i \neq j_{i+1}$ for all $i \in \{1,\cdots,k\}$.[31] Hence:

$$r_{j_i}(i) = r_{j_i}(i+1), \quad \text{for all} \quad i \in \{1,\cdots,k\},$$

and this implies that:

$$\lambda^{j_i}_{r_{j_i}(i)} = \lambda^{j_i}_{r_{j_i}(i+1)}, \quad \text{for all} \quad i \in \{1,\cdots,k\}$$

First case: posteriors have full support. In this case, we should multiply the following candidate equations:

$$\theta_1^{j_k}\lambda^{j_k}_{r_{j_k(1)}} = \theta_1^{j_1}\lambda^{j_1}_{r_{j_1(1)}}, \quad \theta_2^{j_1}\lambda^{j_1}_{r_{j_1(2)}} = \theta_2^{j_2}\lambda^{j_2}_{r_{j_2(2)}}, \quad \cdots\cdots , \quad \theta_k^{j_{k-1}}\lambda^{j_{k-1}}_{r_{j_{k-1}(k)}} = \theta_k^{j_k}\lambda^{j_k}_{r_{j_k(k)}}$$

After canceling all factors $\lambda^{j_i}_{r_{j_i(i)}}$, we obtain the associated cycle equation, that is:

$$\theta_1^{j_k}.\theta_2^{j_1}.\theta_3^{j_2}\cdots\theta_k^{j_{k-1}} = \theta_1^{j_1}.\theta_2^{j_2}.\theta_3^{j_3}\cdots\theta_k^{j_k}$$

For the general case, suppose that there is a state, say $\omega = 1$, and a player, say $j_1 \in J$, such that j_1's posterior that the true state is $\omega = 1$ equals to zero, that is, $\theta_1^{j_1} = 0$. The, we must have $\theta_1^{j_k} = 0$, otherwise players j_1 and j_k cannot have a common prior. But this forces both sides of

$$\theta_1^{j_k}.\theta_2^{j_1}.\theta_3^{j_2}\cdots\theta_k^{j_{k-1}} = \theta_1^{j_1}.\theta_2^{j_2}.\theta_3^{j_3}\cdots\theta_k^{j_k}$$

to be equal to zero. This proves that if the system of candidate equations has a solution, all cycle equations are satisfied, concluding this proof. ■

Proof. (Proposition 2 - Cycle Equations: Sufficiency)

If all cycle equations are satisfied, we will show now that the system has at least one solution. Without loss of generality we may assume that the meet has a unique atom. If this is not the case, the question of rationalizing a common prior is reduced to finding a common prior in each atom δ_m of the meet. As we saw, the assignment of weights to these atoms is completely arbitrary. Thus, for each player $j \in J$, consider the state space decomposition $S = \pi_1^j \cup \pi_2^j \cup \cdots \cup \pi^j_{l(\Pi^j)}$.

If all players have trivial partition $\{S\}$, posteriors coincide with priors and the question of finding a common prior is trivial. Suppose this is not the case, more specifically, assume that player 1 has at least 2 atoms in her knowledge partition. Consider the function $h : [0,+\infty) \to \mathbb{R}$, defined as:

$$h(\overline{\lambda_1^1}) = 1 - \sum_{r=2}^{l(\Pi^j)} \overline{\lambda_r^1},$$

[31] Recall that state $k+1$ is the same as state 1, by definition.

where all values $\overline{\lambda_r^1}$, with $r \in \{2, \cdots, l(\Pi^j)\}$, are functions of $\overline{\lambda_1^1}$ calculated through the use of equations in (4). Since all cycle equations hold, each value $\overline{\lambda_r^1}$ is well defined, regardless of how was its process of calculation (which candidate equation was used first, second,...). Hence, the function $h(\cdot)$ is well defined. It is also decreasing, with $h(0) = 1$ and $\lim_{\overline{\lambda_1^1} \to +\infty} h(\lambda_1^1) = -\infty$. We conclude that $h(\cdot)$ has a unique fixed point. This fixed point, λ_1^1, together with the corresponding values λ_r^1, with $r \in \{2, \cdots, l(\Pi^j)\}$, form a solution to the system (7). Thus, the common prior $\widetilde{\mu}$ may be calculated by (6).

Finally, observe that each variable in the solution must be positive (since a negative or zero value would contaminate all connected component). Because $\sum_{r=1}^{l(\Pi^1)} \lambda_r^1 = 1$, and $\sum_{s=1}^{l(\Pi^2)} \lambda_s^2 = 1$, each λ_r^1 and each λ_s^2 must be at most 1. This concludes this proof. ■

Example 5 *(Multiple Common Priors)*

Suppose that there are 2 *players,* $n = 15$*,* $l(\Pi^1) = 7$*,* $l(\Pi^2) = 6$*,* $l(\Pi^1 \vee \Pi^2) = 11$*,* $l(\Pi^1 \wedge \Pi^2) = 3$ *and posteriors have full support. Then, our system has* 13 *variables and* 16 *candidate equations. Moreover, there are at most* 5 *linearly independent cycle equations. In other words, posteriors can be rationalized by a common prior if and only if **all*** 5 *linearly independent cycle equations hold. If this happens, then there will be only* 11 *linearly independent equations in the system of candidate equations. As we have* 13 *variables, the space of solution will have dimension* 2*. This means that there will be freedom to choose weights* $\rho_1 > 0$*,* $\rho_2 > 0$*,* $\rho_3 > 0$*, with* $\rho_1 + \rho_2 + \rho_3 = 1$*, corresponding to the three atoms of the meet.*

On the other hand, if, for instance, only 4 *linearly independent cycle equations hold, the system of candidate equations will have* 12 *linearly independent equations and* 13 *variables. As we are not accepting zero priors over atoms of the meet, there will be no solution. The freedom given by the existence of more than one atom in the meet cannot compensate a failure of any cycle equation.* ■

Proof. (Proposition 3 - Independent Beliefs Imply Common Prior)

For each $\omega \in S$, denote by $t^j(\omega)$ the type of player j when the true state of the world is ω. Let $\theta^j\left(t^{-j} \mid t^j\right)$ represent j's posterior that other players' types are t^{-j}, when j herself has type t^j. If j has independent beliefs, we denote her posterior about her opponent's type profile simply by $\theta^j\left(t^{-j}\right)$.

Consider the cycle:

$$c = ([\omega_1, \omega_2], [\omega_2, \omega_3], \cdots, [\omega_{k-1}, \omega_k])$$

where $\omega_p = (t^{j_p}, t^{-j_p}) \in S$, for each $p \in \{1, \cdots, k\}$. Consider its corresponding cycle equation:

$$\theta^{j_1}(\omega_1 \mid t^{j_1}(\omega_k)).\theta^{j_2}(\omega_2 \mid t^{j_2}(\omega_1)) \cdots \theta^{j_k}(\omega_k \mid t^{j_k}(\omega_{k-1})) =$$

$$= \theta^{j_k}(\omega_{k-1} \mid t^{j_k}(\omega_k)) \cdots \theta^{j_2}(\omega_1 \mid t^{j_2}(\omega_2)).\theta^{j_1}(\omega_k \mid t^{j_1}(\omega_1))$$

Because beliefs are independent, this can be rewritten as:

$$\theta^{j_1}(\omega_1).\theta^{j_2}(\omega_2) \cdots \theta^{j_k}(\omega_k) = \theta^{j_k}(\omega_{k-1}) \cdots \theta^{j_2}(\omega_1).\theta^{j_1}(\omega_k)$$

But this is true because $\theta^{j_1}(\omega_1) = \theta^{j_1}(\omega_k)$, $\theta^{j_2}(\omega_2) = \theta^{j_2}(\omega_1)$,$\cdots$, $\theta^{j_k}(\omega_k) = \theta^{j_k}(\omega_{k-1})$, by the definition of states ω_p. This proves this proposition. ∎

Proof. (Corollary 1 - Bets)

This is immediate from the necessary and sufficient conditions in Samet (1998a) and by ours, i.e. by proposition 2. ∎

Cournot Duopoly

Let $0 \leq c_L < c_H$, with c_H being sufficiently small so that both players have positive profits in all states. Player 1 of type $t \in \{L, H\}$ chooses quantity q_t^1 such that:

$$\begin{aligned}
q_L^1 &= \arg\max_{q^1} \left\{\theta_1^1 \left[q^1 \left(1 - q^1 - q_L^2 - c_L\right)\right] + \theta_2^1 \left[q^1 \left(1 - q^1 - q_H^2 - c_L\right)\right]\right\} \\
q_H^1 &= \arg\max_{q^1} \left\{\theta_3^1 \left[q^1 \left(1 - q^1 - q_L^2 - c_H\right)\right] + \theta_4^1 \left[q^1 \left(1 - q^1 - q_H^2 - c_H\right)\right]\right\}
\end{aligned}$$

Each type of player 2 has an analogous maximization problem to solve. Because the argument is quadratic, first order conditions are also sufficient. Quantities q_L^1, q_H^1, q_L^2 and q_H^2 are the solution of the linear system:

$$\begin{cases}
2q_L^1 = 1 - c_L - \theta_1^1 q_L^2 - \theta_2^1 q_H^2 \\
2q_H^1 = 1 - c_H - \theta_3^1 q_L^2 - \theta_4^1 q_H^2 \\
2q_L^2 = 1 - c_L - \theta_1^2 q_L^1 - \theta_3^2 q_H^1 \\
2q_H^2 = 1 - c_H - \theta_2^2 q_L^1 - \theta_4^2 q_H^1
\end{cases}$$

Claim 1 $q_L^j > q_H^j$

Proof. Let $X^1 = q_L^1 - q_H^1$ and $X^2 = q_L^2 - q_H^2$. Then, subtracting the second equation from the first and the fourth from the third:

$$\begin{cases}
(\theta_1^2 - \theta_2^2)X^1 + 2X^2 = c_H - c_L \\
2X^1 + (\theta_4^1 - \theta_2^1)X^2 = c_H - c_L
\end{cases}$$

Solving this system:

$$X^1 = \frac{\left[2 - (\theta_4^1 - \theta_2^1)\right](c_H - c_L)}{4 - (\theta_4^1 - \theta_2^1)(\theta_1^2 - \theta_2^2)}, \qquad \text{and} \qquad X^2 = \frac{\left[2 - (\theta_1^2 - \theta_2^2)\right](c_H - c_L)}{4 - (\theta_4^1 - \theta_2^1)(\theta_1^2 - \theta_2^2)}$$

Because $X^1 > 0$ and $X^2 > 0$, we conclude that $q_L^1 > q_H^1$ and $q_L^2 > q_H^2$. This proves claim 1. ∎

Claim 2 *For each fixed type of player* 2*, the expected profit of player* 1 *is higher when player* 1 *herself is low type.*

Proof. Let $P^j(\omega)$ denote the profit of player j at state ω. After using the first order conditions and simplifying, we find that the difference between the expected profit of a low type player 1 and the expected profit of a high type of player 1, both facing a low type of player 2 is:

$$E^1\left[P^1(LL)\right] - E^1\left[P^1(HL)\right] = (q_L^1)^2 - (q_H^1)^2,$$

which is positive because $X^1 > 0$. A similar result holds when player 2 is a high type and also for the expected profits of player 2. This proves claim 2. ■

Claim 3 *For each fixed type of player* 2*, the profit of a low type of player* 1 *is larger than the profit of a high type of player* 1*.*

Proof. To see this observe that:

$$2P^1(LL) - 2P^1(HL) = 2q_L^1\left(1 - c_L - q_L^1 - q_L^2\right) - 2q_H^1\left(1 - c_H - q_H^1 - q_L^2\right) =$$

$$= q_L^1\left(\theta_1^1 q_L^2 + \theta_2^1 q_H^2 + \theta_1^2 q_L^1 + \theta_3^2 q_H^1\right) - q_H^1\left(\theta_3^1 q_L^2 + \theta_4^1 q_H^2 + \theta_1^2 q_L^1 + \theta_3^2 q_H^1 - c_L - c_H\right) =$$

$$= \left(\theta_1^2 q_L^1 + \theta_3^2 q_H^1\right) X^1 + q_L^1\left(\theta_1^1 q_L^2 + \theta_2^1 q_H^2\right) - q_H^1\left(\theta_3^1 q_L^2 + \theta_4^1 q_H^2 + c_L - c_H\right)$$

Since $\left(\theta_1^2 q_L^1 + \theta_3^2 q_H^1\right) X^1 > 0$ and $q_L^1 > q_H^1$, to prove that $P^1(LL) > P^1(HL)$ it is enough to prove that $\theta_1^1 q_L^2 + \theta_2^1 q_H^2 > \theta_3^1 q_L^2 + \theta_4^1 q_H^2 + c_L - c_H$. But, after some algebra, this is equivalent to show that $c_H - c_L > X^1$, which always holds.[32] Similarly, we can prove that $P^1(LH) > P^1(HH)$, $P^2(LL) > P^2(LH)$ and $P^2(HL) > P^2(HH)$. This concludes this proof. ■

Claim 4 $P^1(LH) > P^1(LL)$ *and* $P^1(HH) > P^1(HL)$*. In words, regardless of her own type, a player has more profit when her opponent is a high type.*

Proof. Because $P^1(LH) = q_L^1\left(1 - q_L^1 - q_H^2 - c_L\right)$ and $P^1(LH) = q_L^1\left(1 - q_L^1 - q_H^2 - c_L\right)$ a direct calculation reveals that:

$$P^1(LH) > P^1(LL) \Leftrightarrow q_L^2 > q_H^2,$$

but claim 1 proves the last inequality. A similar argument shows that $P^1(HH) > P^1(HL)$. This proves this claim. ■

Characterizing Bets on the Duopoly Example

Let $\lambda \in \left(\frac{\theta_2^1\theta_4^2}{\theta_1^1\theta_2^2}, \frac{\theta_4^1\theta_3^2}{\theta_3^1\theta_1^2}\right)$ be arbitrary. Suppose that $Opt^1 < Opt^2$. Note that $\frac{\theta_4^1\theta_3^2}{\theta_3^1\theta_1^2} > \frac{\theta_2^1\theta_4^2}{\theta_1^1\theta_2^2}$, so there are many possible values for λ. Let one of the values g_1 or g_4 be an arbitrary positive number, and the other be such that $g_1 = \lambda g_4$. Let $g_2 \in \left(\frac{-\theta_1^1 \cdot g_1}{\theta_2^1}, \frac{-\theta_4^2 \cdot g_4}{\theta_2^2}\right)$,

[32] In fact, it not hard to see that $c_H - c_L \geq 4X^1 > 0$.

and $g_3 \in \left(\frac{-\theta_4^1 \cdot g_4}{\theta_3^1}, \frac{-\theta_1^2 \cdot g_1}{\theta_3^2}\right)$ be arbitrary. Again, $\frac{-\theta_4^2 \cdot g_4}{\theta_2^2} > \frac{-\theta_1^1 \cdot g_1}{\theta_2^1}$ and $\frac{-\theta_1^2 \cdot g_1}{\theta_3^2} > \frac{-\theta_4^1 \cdot g_4}{\theta_3^1}$ because $Opt^1 < Opt^2$, regardless of all previous choices. ■

Bets on a Cycle - Two-Player Games

Consider the cycle:

$$c = ([1,2],[2,3],[3,4],\cdots,[2k-1,2k]),$$

where k is a positive integer number. Generalize the definition of players' optimism in the following way:

$$Opt^1 = \frac{\theta_2^1 . \theta_4^1 \cdots \theta_{2k}^1}{\theta_1^1 . \theta_3^1 \cdots \theta_{2k-1}^1}, \qquad \text{and} \qquad Opt^2 = \frac{\theta_2^2 . \theta_4^2 \cdots \theta_{2k}^2}{\theta_1^2 . \theta_3^2 \cdots \theta_{2k-1}^2}$$

Define the transfers g_ω^j as before. The expected profits of players 1 and 2, as well as player 3 (market maker) profits are:

$$\begin{aligned}
E^1 [G(\alpha_r)] &= \theta_{2r-1}^1 . g_{2r-1}^1 + \theta_{2r}^1 . g_{2r}^1, \qquad \forall r \in \{1,\cdots,k\} \\
E^2 [-G(\beta_s)] &= \theta_{2s}^2 . g_{2s}^2 + \theta_{2s+1}^2 . g_{2s+1}^2, \qquad \forall s \in \{1,\cdots,k\} \\
\text{where} \quad \theta_{2k+1}^2 &= \theta_1^2, \qquad \text{and} \qquad g_{2k+1}^2 = g_1^2 \\
g_\omega^3 &= g_\omega^2 - g_\omega^1, \qquad\qquad \forall \omega \in S
\end{aligned}$$

If $Opt^1 = Opt^2$, there is no bet. If $Opt^1 < Opt^2$, then, a feasible bet, i.e. a bet in which it is always common knowledge that all players have positive expected gains, is such that player 1 wins in states $\omega \in \{1,3,5,\cdots,2k-1\}$ and loses in states $\omega \in \{2,4,6,\cdots,2k\}$. If $Opt^1 > Opt^2$, player 1 wins in states $\omega \in \{2,4,6,\cdots,2k\}$ and loses in states $\omega \in \{1,3,5,\cdots,2k-1\}$.

Normalize monetary units such that $g_1^1 = 1$. The associated system has $4k$ equations and $4k-1$ variables. This extra degree of freedom imposes a relation among all g_ω^3, which is similar to (14). The constant term is similar to Δ, and its sign will be the same as the sign of $Opt^2 - Opt^1$.

Choosing all $E^1 [G(\alpha_r)]$ and $E^2 [-G(\beta_s)]$ sufficiently close to zero, player 3 has a risk free profit, that is, she can find all $g_\omega^3 > 0$. ■

Cycles with 3 Players

We conclude with the following example. It shows how to construct a bet in a cycle with 3 edges. Consider the model $\left(S, \left\{(\Pi^1,\theta^1),(\Pi^2,\theta^2),(\Pi^3,\theta^3)\right\}\right)$, with $S = \{a,b,c\}$, and such that:

$$\Pi^1 = \{\{a,b\},\{c\}\}, \quad \Pi^2 = \{\{b,c\},\{a\}\}, \quad \Pi^3 = \{\{c,a\},\{b\}\}$$

Observe that in any sub-model with only 2 players it is always possible to find a common prior because there is no cycle. Thus, player 1, for instance, will not be able to explore (from her perspective), via a bet, an eventual difference in beliefs with player 2, unless player 3 comes along.

Consider the model with all players now. Then, posteriors may be rationalized by a common prior if and only if the following cycle equation holds:

$$\theta_a^1.\theta_b^2.\theta_c^3 = \theta_b^1.\theta_c^2.\theta_a^3 \tag{15}$$

Now, assume that posteriors cannot be rationalized by a common prior. Hence, by proposition 2, the cycle equation (15) does not hold. Consider the following bet (lottery G) among the players. In state $\omega = a$, player 3 pays $g_a > 0$ to player 1. In state $\omega = b$, player 1 pays $g_b > 0$ to player 2. Finally, at $\omega = c$, player 2 pays $g_c > 0$ to player 3. The expected profits of players 1, 2 and 3 are either zero (in states $\omega = c$, a, b, respectively), or e^1, e^2, e^3, respectively, where:

$$\begin{aligned} e^1 &= +\theta_a^1.g_a - \theta_b^1.g_b \\ e^2 &= +\theta_b^2.g_b - \theta_c^2.g_c \\ e^3 &= +\theta_c^3.g_c - \theta_a^3.g_a \end{aligned}$$

Again, we may obtain $e^1 > 0$, $e^2 > 0$ and $e^3 > 0$, as long as $\theta_a^1.\theta_b^2.\theta_c^3 > \theta_b^1.\theta_c^2.\theta_a^3$.[33] The expected payoffs hiper-plane is:

$$\frac{e^1}{\left(\frac{\theta_b^1\theta_c^2\theta_a^3}{\theta_b^2\theta_c^3}\right)} + \frac{e^2}{\left(\frac{\theta_c^2\theta_a^3}{\theta_c^3}\right)} + \frac{e^3}{\left(\theta_a^3\right)} = \frac{\theta_a^1\theta_b^2\theta_c^3 - \theta_b^1\theta_c^2\theta_a^3}{\theta_b^1\theta_c^2\theta_a^3}$$

We conclude that when players 1, 2 and 3 are present, each one of them can explore the differences in beliefs via the proposed bet whenever the cycle equation (15) does not hold. ■

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